

DERIVATIVES (examples - part I)

The primary operation in differential calculus is finding a derivative.

Table of derivatives

1. $C' = 0$

2. $x' = 1$

3. $(x^2)' = 2x$

4. $(x^n)' = nx^{n-1}$

5. $(a^x)' = a^x \ln a$

6. $(e^x)' = e^x$

7. $(\log_a x)' = \frac{1}{x \ln a}$ (here is $x > 0$ and $a > 0$)

8. $(\ln x)' = \frac{1}{x}$ ($x > 0$)

9. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ ($x \neq 0$)

10. $\sqrt{x}' = \frac{1}{2\sqrt{x}}$ ($x > 0$)

11. $(\sin x)' = \cos x$

12. $(\cos x)' = -\sin x$

13. $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ $x \neq \frac{\pi}{2} + k\pi$

14. $(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$ $x \neq k\pi$

15. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $|x| < 1$

16. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

17. $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

18. $(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$

General differentiation rules:

1. $[cf(x)]' = cf'(x)$

2. $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

} **Linearity**

3. $(u \cdot v)' = u'v + v'u$ **Product rule**

4. $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$ **Quotient rule**

Examples:

1. Find derivatives for following functions:

a) $y = x^5$

b) $y = 10^x$

c) $f(x) = \sqrt{x}$

d) $y = \log_3 x$

e) $f(x) = \sqrt[3]{x^5}$

f) $f(x) = \frac{1}{x^7}$

g) $y = \frac{1}{\sqrt[8]{x^5}}$

h) $y = x\sqrt{x}$

i) $y = \frac{x^2\sqrt{x}}{\sqrt[3]{x^2}}$

Solution:

a) $y = x^5 \Rightarrow y' = 5x^4$ as the 4. in table

b) $y = 10^x \Rightarrow y' = 10^x \ln 10$ as the 5. in table

c) $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$ as the 10. in table

d) $y = \log_3 x$ so: $y' = \frac{1}{x \ln 3}$ as the 7. in table

e) $f(x) = \sqrt[3]{x^5}$ Look out: Here first we have to "prepare" function for derivative: We will use: $\sqrt[n]{x^n} = x^{\frac{n}{m}}$. So:
 $\sqrt[3]{x^5} = x^{\frac{5}{3}}$ and still working as $(x^n)' = nx^{n-1} \longrightarrow f'(x) = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}}$

f) $f(x) = \frac{1}{x^7}$ And here we must "prepare" function. How is $\frac{1}{a^n} = a^{-n}$ it is $\frac{1}{x^7} = x^{-7}$ and $f'(x) = -7x^{-7-1} = -7x^{-8}$

g) $y = \frac{1}{\sqrt[8]{x^5}} \longrightarrow y = x^{-\frac{5}{8}} \longrightarrow y' = -\frac{5}{8}x^{-\frac{5}{8}-1} = -\frac{5}{8}x^{-\frac{13}{8}}$

h) $y = x\sqrt{x} = x^1x^{\frac{1}{2}} = x^{\frac{3}{2}} \longrightarrow y' = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

i) $y = \frac{x^2\sqrt{x}}{\sqrt[3]{x^2}} = \frac{x^2x^{\frac{1}{2}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{5}{2}}}{x^{\frac{2}{3}}} = x^{\frac{11}{6}} \longrightarrow y' = \frac{11}{6}x^{\frac{11}{6}-1} = \frac{11}{6}x^{\frac{5}{6}}$

2. Find derivatives for following functions:

a) $y = 5 \sin x$

b) $y = \frac{1}{2} \ln x$

c) $y = \frac{-\sqrt{3}}{4} \operatorname{tg} x$

d) $y = \pi x^3$

e) $f(x) = \frac{4}{5} \operatorname{arctg} x$

f) $f(x) = -a \operatorname{ctg} x$

g) $y = 10$

h) $y = -2abx$

Solution:

a) $y = 5 \sin x$ rule: $[cf(x)]' = cf'(x)$

$y' = 5 \cos x$

b) $y = \frac{1}{2} \ln x \longrightarrow y' = \frac{1}{2} \frac{1}{x} = \frac{1}{2x}$

c) $y = \frac{-\sqrt{3}}{4} \operatorname{tg} x \longrightarrow y' = \frac{-\sqrt{3}}{4} \frac{1}{\cos^2 x}$

d) $y = \pi x^3$ Take heed: π is also constant... $y' = \pi 3x^2$

$$e) \quad f(x) = \frac{4}{5} \arctg x \quad \longrightarrow \quad f'(x) = \frac{4}{5} \frac{1}{1+x^2} = \frac{4}{5(1+x^2)}$$

$$f) \quad f(x) = -a \operatorname{ctg} x \quad \longrightarrow \quad f'(x) = -a \left(-\frac{1}{\sin^2 x} \right) = \frac{a}{\sin^2 x}$$

$$g) \quad y = 10 \quad \longrightarrow \quad y' = 0$$

$$h) \quad y = -2abx \quad \longrightarrow \quad y' = -2ab$$

3. Find derivatives for following functions:

$$a) \quad y = 5x^6 - 3x^5 + 4x - 8$$

$$b) \quad f(x) = 3\sin x - \frac{1}{2}e^x + 7\arctg x - 5$$

$$c) \quad y = \sqrt[3]{x} - \frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{5x^3} + 4$$

Solution:

$$a) \quad y = 5x^6 - 3x^5 + 4x - 8 \quad \text{use rule: } [f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

$$y' = 5(x^6)' - 3(x^5)' + 4(x)' - 8'$$

$$y' = 30x^5 - 15x^4 + 4 - 0$$

$$y' = 30x^5 - 15x^4 + 4$$

$$b) \quad f(x) = 3\sin x - \frac{1}{2}e^x + 7\arctg x - 5$$

$$f'(x) = 3(\sin x)' - \frac{1}{2}(e^x)' + 7(\arctg x)' - 5'$$

$$f'(x) = 3 \cos x - \frac{1}{2}e^x + 7 \frac{1}{1+x^2} - 0 = 3 \cos x - \frac{1}{2}e^x + \frac{7}{1+x^2}$$

$$c) \quad y = \sqrt[3]{x} - \frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{5x^3} + 4 \quad \text{we have to "prepare" function for derivative.....}$$

$$y = x^{\frac{1}{3}} - 2x^{-\frac{1}{2}} + 3x^{-2} - \frac{1}{5}x^{-3} + 4$$

$$y' = \frac{1}{3}x^{-\frac{2}{3}} - 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + 3(-2)x^{-3} - \frac{1}{5}(-3)x^{-4} + 0 = \frac{1}{3}x^{-\frac{2}{3}} + x^{-\frac{3}{2}} - 6x^{-3} + \frac{3}{5}x^{-4}$$

4. Find derivatives:

a) $f(x) = x^3 \sin x$

b) $f(x) = e^x \arcsin x$

c) $y = (3x^2+1)(2x^2+3)$

d) $y = x - \sin x \cos x$

Solution:

As you noticing, in this task, we must use the rule for derivative products: $(\mathbf{u} \circ \mathbf{v})' = \mathbf{u}' \mathbf{v} + \mathbf{v}' \mathbf{u}$

a) $f(x) = x^3 \sin x$ Here is $x^3 \longleftrightarrow u$ and $\sin x \longleftrightarrow v$

$$f'(x) = (x^3)' \sin x + (\sin x)' x^3$$

$$f'(x) = 3x^2 \sin x + \cos x x^3 = x^2(3 \sin x + x \cos x)$$

b) $f(x) = e^x \arcsin x$ $e^x \longleftrightarrow u$ and $\arcsin x \longleftrightarrow v$

$$f'(x) = (e^x)' \arcsin x + (\arcsin x)' e^x$$

$$f'(x) = e^x \arcsin x + \frac{1}{\sqrt{1-x^2}} e^x = e^x \left(\arcsin x + \frac{1}{\sqrt{1-x^2}} \right)$$

c) $y = (3x^2+1)(2x^2+3)$

$$y' = (3x^2+1)'(2x^2+3) + (3x^2+1)(2x^2+3)' = 6x(2x^2+3) + 4x(3x^2+1) = 2x[(6x^2+9) + (6x^2+2)] = 2x[12x^2+11]$$

d) $y = x - \sin x \cos x$

$$y' = 1 - [(\sin x)' \cos x + (\cos x)' \sin x]$$

$$y' = 1 - [\cos x \cos x - \sin x \sin x] \quad \text{We know that : } \sin^2 x + \cos^2 x = 1$$

$$y' = \sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x = 2 \sin^2 x$$

5. Find derivatives for following functions:

a) $y = \frac{x^2 + 1}{x^2 - 1}$

b) $y = \frac{\cos x}{1 - \sin x}$

c) $y = \frac{5 - e^x}{e^x + 2}$

d) $y = \frac{\ln x + 1}{\ln x}$

Solution:

Here we use quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

a) $y = \frac{x^2 + 1}{x^2 - 1}$ here is $x^2 + 1 \longrightarrow u$ and $x^2 - 1 \longrightarrow v$

$y' = \frac{(x^2 + 1)'(x^2 - 1) - (x^2 - 1)'(x^2 + 1)}{(x^2 - 1)^2}$ denominator shall remain so until the end!

$y' = \frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 - 1)^2}$

$y' = \frac{2x[(x^2 - 1) - (x^2 + 1)]}{(x^2 - 1)^2}$ simplify little ...

$y' = \frac{-4x}{(x^2 - 1)^2}$ here is the final solution!

$$\mathbf{b)} \quad y = \frac{\cos x}{1 - \sin x}$$

$$y' = \frac{(\cos x)'(1 - \sin x) - (1 - \sin x)'\cos x}{(1 - \sin x)^2}$$

$$y' = \frac{-\sin x(1 - \sin x) + \cos x \cos x}{(1 - \sin x)^2}$$

$$y' = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} \quad \text{as is: } \sin^2 x + \cos^2 x = 1 \quad \text{it is:}$$

$$y' = \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$y' = \frac{1}{1 - \sin x} \quad \mathbf{\text{final solution}}$$

$$\mathbf{c)} \quad y = \frac{5 - e^x}{e^x + 2}$$

$$y' = \frac{(5 - e^x)'(e^x + 2) - (e^x + 2)'(5 - e^x)}{(e^x + 2)^2}$$

$$y' = \frac{-e^x(e^x + 2) - e^x(5 - e^x)}{(e^x + 2)^2}$$

$$y' = \frac{-e^x(e^x + 2 + 5 - e^x)}{(e^x + 2)^2} \quad \text{simplify little ...}$$

$$y' = \frac{-7e^x}{(e^x + 2)^2}$$

$$d) \quad y = \frac{\ln x + 1}{\ln x}$$

$$y' = \frac{(\ln x + 1)' \ln x - (\ln x)' (\ln x + 1)}{\ln^2 x}$$

$$y' = \frac{\frac{1}{x} \ln x - \frac{1}{x} (\ln x + 1)}{\ln^2 x}$$

$$y' = \frac{\frac{1}{x} \ln x - \frac{1}{x} \ln x - \frac{1}{x}}{\ln^2 x}$$

$$y' = \frac{-\frac{1}{x}}{\ln^2 x} \quad \text{so: } y' = \frac{-1}{x \ln^2 x} \quad \text{is the final solution.}$$

6. Determine the equation of tangent for function $y = 2x^2 - 3x + 2$ in the point A (2, y), which belongs to the function.

Solution:

First, we find unknown y ; we will replace $x = 2$ in the function ...

$$y = 2 * 2^2 - 6 + 2 = 4, \quad \text{and the point is actually } A(2, 4)$$

To remind you:

$$\text{Tangent equation is: } y - y_0 = f'(x_0)(x - x_0)$$

$$f(x) = 2x^2 - 3x + 2 \quad \text{we find derivate...}$$

$$f'(x) = 4x - 3 \quad \text{replace } x = 2$$

$$f'(2) = 8 - 3 = 5$$

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y - 4 = 5(x - 2) \quad \text{simplify little ...}$$

$$y = 5x - 6 \quad \text{is requested equation of tangent}$$

7. In which point on parabola $y = x^2 - 7x + 3$ is tangent line parallel with line $y = 5x + 2$?

Solution:

$$f(x) = x^2 - 7x + 3$$

$$f'(x) = 2x - 7$$

Condition that two lines are parallel is $k_1 = k_2$, from $y = 5x + 2$ is $k = 5$, so: $f'(x) = 5$

$$2x - 7 = 5$$

$$2x = 12$$

$$x = 6$$

Now this value change in the equation of parabola to find y. So:

$$y = x^2 - 7x + 3$$

$$y = 36 - 42 + 3$$

$$y = -3$$

Requested point, which belongs to the parabola is: (6, -3).

8. Determine the equation of normal line for function $y = x^4 - x^2 + 3$ in point M (1, y), which belongs to function .

Solution:

First, find unknown coordinate y.

$$y = 1 - 1 + 3 = 3, \text{ so coordinates are } M(1, 3)$$

$$y - y_0 = \frac{-1}{f'(x_0)} (x - x_0) \quad \text{equation of normal line}$$

$$y = x^4 - x^2 + 3$$

$$y' = 4x^3 - 2x$$

$$y'(1) = 4 - 2 = 2$$

$$y - 3 = \frac{-1}{2}(x - 1) \text{ simplify little ...}$$

$$2y - 6 = -x + 1 \longrightarrow n : x + 2y - 7 = 0 \text{ is requested solution}$$